

Multi-Mass Scale RG Improvement of the Coleman-Weinberg Model

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(Dated: February 2, 2008)

We obtain the RG improvement of the effective potential for the Coleman-Weinberg model by resumming the leading-logarithms which have three different mass scales. Then we investigate the effect of the multi-mass scale on the prediction of the magnitude of the Higgs boson mass by considering the two-loop effective potential.

PACS numbers: 11.10.Hi, 11.15.Bt, 11.30.Qc

I. INTRODUCTION

The Coleman-Weinberg (CW) model [1] is the massless scalar electrodynamics where the scalar field does not have a tree level mass and the spontaneous symmetry breaking occurs from the effective potential [2] which is the radiative correction to the classical potential. The CW model and its extension to the more realistic model have been studied extensively due to its predictive power for the magnitude of the Higgs boson mass. The leading logarithms of the effective potential can be resummed by using the renormalization group(RG) [3] known as RG improvement and recently, the RG improvement of the effective potential of the CW model has been obtained by the optimal form [4] which incorporates all possible logarithms of single mass scale to the effective potential that is accessible via RG methods.

However, actually the effective potential of the CW model has three different mass scales and in this paper, we will obtain the complete RG improvement of the leading- logarithms of the effective potential for the CW model by using the method of RG improvement in case of the multi-mass scale [5]. Then we will investigate the prediction of the Higgs boson mass by using the two-loop effective potential where the difference between the case of single mass scale and that of multi-mass scale appears for the first time.

II. RG IMPROVEMENT OF THE CW MODEL

In this section, we will first obtain the RG improvement of the CW model by using the method of the RG improvement in case of the multi-mass scale. The classical Lagrangian of the CW model is given by

$$L = \frac{1}{2}(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + \frac{1}{2}(\partial_\mu\phi_2 + eA_\mu\phi_1)^2 - \frac{\lambda}{24}(\phi_1^2 + \phi_2^2). \quad (1)$$

In this paper, we will use the parameters x and y defined by

$$x \equiv \frac{e^2}{4\pi^2} \quad \text{and} \quad y \equiv \frac{\lambda}{4\pi^2} \quad (2)$$

The effective potential of the CW model is independent of the renormalization mass scale μ and hence satisfies the renormalization group equation

$$[\mu \frac{\partial}{\partial \mu} + \beta_x \frac{\partial}{\partial x} + \beta_y \frac{\partial}{\partial y} + \gamma \phi \frac{\partial}{\partial \phi}] V_{eff} = 0 \quad (3)$$

where

$$\phi^2 = \phi_1^2 + \phi_2^2 \quad (4)$$

and the RG functions β and γ are given by

$$\beta_f = \mu \frac{df}{d\mu} = \kappa \beta_f^{(1)} + \kappa^2 \beta_f^{(2)} + \dots (f = x, y) \quad (5)$$

and

$$\gamma = \frac{\mu d\phi}{\phi d\mu} = \kappa \gamma_f^{(1)} + \kappa^2 \gamma_f^{(2)} + \dots \quad (6)$$

with $\kappa = (4\pi^2)^{-1}$.

By using the method of characteristics[6], we can see that the effective potential satisfies

$$V(x, y, \phi, \mu) = V(x(t), y(t), \phi(t), \mu(t)) \quad (7)$$

where

$$\mu(t) = \mu e^t \quad (8)$$

and $x(t), y(t)$ and $\phi(t)$ is the solution of the differential equation such that

$$\frac{df(t)}{dt} = \beta_f(x(t), y(t)) = \kappa \beta_f^{(1)}(x(t), y(t)) + \kappa^2 \beta_f^{(2)}(x(t), y(t)) + \dots (f = x, y) \quad (9)$$

and

$$\frac{d\phi(t)}{dt} = \gamma(x(t), y(t))\phi(t) = \kappa \gamma_f^{(1)}(x(t), y(t)) + \kappa^2 \gamma_f^{(2)}(x(t), y(t)) + \dots \quad (10)$$

with the initial conditions $x(0) = x, y(0) = y$ and $\phi(0) = \phi$.

Since the CW model is a O(2) scalar field theory coupled to the U(1) gauge field, the effective potential of the CW model contains the following three different mass scales.

$$L_1 \equiv \log\left(\frac{2\pi^2 y\phi^2}{\mu^2}\right), L_2 \equiv \log\left(\frac{2\pi^2 y\phi^2}{3\mu^2}\right) \text{ and } L_3 \equiv \log\left(\frac{4\pi^2 x\phi^2}{\mu^2}\right). \quad (11)$$

The resummation of these three different leading-logarithms can be done by following the similar steps to the case of two different mass scales[5] as follows. First let us write the effective potential of the CW model as

$$V(x, y, \phi, \mu) = \frac{\pi^2}{6} y\phi^4 + \sum_{l=1}^{\infty} \kappa^l \phi^4 \sum_{n=0}^l \sum_{\substack{p,q,r \\ p+q+r=n}} f^{(l,n,p,q,r)}(x, y) L_1^p L_2^q L_3^r \quad (12)$$

where l is the loop order and the case $n = l$ ($n = l - 1$) corresponds to the leading- (next-to-leading) logarithms etc.

By noting that one can choose arbitrary value for t in Eqs.(7-10), let us rescale the variables κ, t and the mass scales $L_i (i = 1, 2, 3)$ at both sides of these equations as

$$\kappa \rightarrow \hbar\kappa, \quad L_i \rightarrow \frac{L_i}{\hbar} \quad \text{and} \quad t \rightarrow \frac{t}{\hbar} \quad (13)$$

and then substitute the effective potential given in Eq.(12) into Eq.(7). Since the leading logarithms of the effective potential given in L.H.S. of Eq.(12) does not change under this rescaling, we can obtain the resummation of the leading-logarithms (V_{LL}) by taking the order \hbar^0 terms of the R.H.S. of Eq.(7) with the effective potential given in Eq.(12). Then we obtain

$$V_{LL} = \frac{\pi^2}{6} y_0(t) \phi_0(t)^4 + \sum_{l=1}^{\infty} \kappa^l \phi_0(t)^4 \sum_{\substack{p,q,r \\ p+q+r=l}} f^{(l,l,p,q,r)}(x_0(t), y_0(t)) \times (L_1 - 2t)^p (L_2 - 2t)^q (L_3 - 2t)^r \quad (14)$$

Here the quantities with the subscript 0 are the order \hbar^0 solutions of the Eq.(9) and Eq.(10) under the rescaling of Eq.(13) so that

$$\frac{df_0(t)}{dt} = \kappa \beta_f^{(1)}(x_0(t), y_0(t)) \quad (f = x, y) \quad (15)$$

and

$$\frac{d\phi_0(t)}{dt} = \kappa \gamma^{(1)}(x_{(0)}(t), y_{(0)}(t)) \phi_{(0)}(t) \quad (16)$$

Since we can choose arbitrary value for the variable t , we choose

$$t = \frac{L_3}{2} \quad (17)$$

Then only those terms with $r = 0$ in Eq.(14) survives and we obtain

$$V_{LL} = \frac{\pi^2}{6} y_0\left(\frac{L_3}{2}\right) \phi_0\left(\frac{L_3}{2}\right)^4 + \sum_{l=1}^{\infty} \kappa^l \phi_0\left(\frac{L_3}{2}\right)^4 \sum_{\substack{p,q \\ p+q=l}} f^{(l,l,p,q,0)}(x_0\left(\frac{L_3}{2}\right), y_0\left(\frac{L_3}{2}\right)) (L_1 - L_3)^p (L_2 - L_3)^q \quad (18)$$

Since the contributions from the coupling with gauge fields ($r \neq 0$ term in Eq.(14)) has vanished, this is nothing but the resummation of the leading-logarithms of the effective potential for the O(2) scalar field theory with coupling constants $f_0\left(\frac{L_3}{2}\right)$ ($f = x, y$), classical field $\phi_0\left(\frac{L_3}{2}\right)$ and the mass scales $(L_1 - L_3)$ and $(L_2 - L_3)$. By using the results given in [5] we obtain

$$V_{LL} = \frac{\pi^2}{6} \frac{y_0\left(\frac{L_3}{2}\right) \phi_0\left(\frac{L_3}{2}\right)^4}{1 - \frac{3}{8} y_0\left(\frac{L_3}{2}\right) (L_1 - L_3) - \frac{1}{24} y_0\left(\frac{L_3}{2}\right) (L_2 - L_3)} \quad (19)$$

In order to obtain the coupling constants $f_0\left(\frac{L_3}{2}\right)$ ($f = x, y$) and the classical field $\phi_0\left(\frac{L_3}{2}\right)$, we use the one-loop RG functions of the CW model [1] to obtain

$$\frac{dx_0(t)}{dt} = \beta_x^{(1)} x_0(t)^2 = \frac{1}{6} x_0(t)^2 \quad (20)$$

$$\frac{d\phi_0(t)}{dt} = \gamma^{(1)} x_0(t) \phi_0(t) = \frac{3}{4} x_0(t) \phi_0(t) \quad (21)$$

and

$$\frac{dy_0(t)}{dt} = \beta_y^{(1)} y_0(t)^2 + \beta_{yx}^{(1)} x_0(t) y_0(t) + \beta_{yxx}^{(1)} x_0(t)^2 = \frac{5}{6} y_0(t)^2 - 3 x_0(t) y_0(t) + 9 x_0(t)^2 \quad (22)$$

Eqs.(20) and (21) can be solved easily and we can obtain

$$x_0(t) = \frac{x}{1 - \frac{1}{6} xt} \quad (23)$$

$$\phi_0(t) = \frac{\phi}{(1 - \frac{1}{6} xt)^{9/2}} \quad (24)$$

In order to obtain the solution of the Eq.(22), we write $y_0(t)$ as [7]

$$y_0(t) = -\frac{F'(t)}{\beta_y^{(1)} F(t)} \quad (25)$$

where $F(t)$ is an auxiliary function.

By substituting this expression into Eq.(22), we obtain

$$F''(t) - \beta_{yx}^{(1)}x(t)F'(t) + \beta_y^{(1)}\beta_{yxx}^{(1)}x(t)^2F(t) = 0 \quad (26)$$

This is a Euler differential equation[8] and by changing the variable from t to z as

$$z = \frac{\beta_y^{(1)}}{\beta_x^{(1)}} \log(1 - \beta_x^{(1)}xt) \quad (27)$$

we obtain

$$\beta_y^{(1)} \frac{d^2F(z)}{dz^2} + (\beta_{yx}^{(1)} - \beta_y^{(1)}) \frac{dF(z)}{dz} + \beta_{yxx}^{(1)}F(z) = 0 \quad (28)$$

By solving this equation and by using the initial condition for $y(t)$ as $y(0) = y$, we finally obtain

$$y_0(t) = x \frac{a(y - bx)G(t)^{a\delta-1} - b(y - ax)G(t)^{b\delta-1}}{(y - bx)G(t)^{a\delta} - (y - ax)G(t)^{b\delta}} \quad (29)$$

where $\delta \equiv \beta_y^{(1)}/\beta_x^{(1)}$ and

$$G(t) \equiv \frac{x}{x(t)} = 1 - \frac{1}{6}xt \quad (30)$$

and a and b are the two roots of the equation

$$\beta_y^{(1)}p^2 + (\beta_{yx}^{(1)} - \beta_x^{(1)})p + \beta_{yxx}^{(1)} = 0 \quad (31)$$

This result agrees with the one given in Ref. [5] obtained by some other method. By using Eq.(20) and Eq.(22), can obtain

$$\frac{dR(t)}{dt} = x(t)[\beta_y^{(1)}R(t)^2 + (\beta_{yx}^{(1)} - \beta_x^{(1)})R(t) + \beta_{yxx}^{(1)}] \quad (32)$$

where $R(t)$ is the ratio between the two parameters x and y defined by

$$R(t) = \frac{y_0(t)}{x_0(t)} \quad (33)$$

Then we can see that the two roots a and b of Eq.(31) becomes the fixed point[9] of the ratio $R(t)$.

By substituting the coefficients of the one-loop RG functions of the CW model given in Eq(20) and Eq.(22), we obtain the two roots a and b of Eq.(31) as

$$a, b = \frac{19}{10} \pm i \frac{\sqrt{719}}{10} \quad (34)$$

By substituting the two roots a and b into Eq.(29), we obtain $y_0(t)$ as

$$y_0(t) = \frac{x}{G(t)} \frac{y + \frac{1}{\sqrt{719}}(19y - 108x)\tan(\frac{\sqrt{719}}{2}\ln G(t))}{x - \frac{1}{\sqrt{719}}(19x - 10y)\tan(\frac{\sqrt{719}}{2}\ln G(t))} \quad (35)$$

By substituting Eq.(24) and (35) into Eq.(19) we can obtain the resummation of the leading-logarithms terms of the effective potential for the CW model. In order to compare with the perturbative expansion of the optimal RG improvement given in [4] where single mass scale was considered, let us substitute the same value for three different mass scale as $L_1 = L_2 = L_3 = L$ into the resummation of the leading-logarithms given in Eq.(19) where

$$L \equiv \log(\frac{\phi^2}{\mu^2}) \quad (36)$$

Then we obtain

$$V_{LL} = \frac{\pi^2 x \phi^4}{6 G(\frac{L}{2})^{19}} \frac{y + \frac{1}{\sqrt{719}}(19y - 108x) \tan(\frac{\sqrt{719}}{2} \ln G(\frac{L}{2}))}{x - \frac{1}{\sqrt{719}}(19x - 10y) \tan(\frac{\sqrt{719}}{2} \ln G(\frac{L}{2}))} \quad (37)$$

Now we should expand the functions appearing in Eq.(37) as a power series in x . By using Eq.(30), the expansion as a power series in x of the $G(\frac{L}{2})^{19}$ in the denominator of above equation is straightforward and $\frac{1}{\sqrt{719}} \tan(\frac{\sqrt{719}}{2} \ln G(\frac{L}{2}))$ can be expanded as a power series in x as

$$\frac{1}{\sqrt{719}} \tan(\frac{\sqrt{719}}{2} \ln G(\frac{L}{2})) = -\frac{1}{24} Lx - \frac{1}{576} L^2 x^2 - \frac{241}{13824} L^3 x^3 + O(x^4) \quad (38)$$

By substituting these results into Eq.(37) and writing the power series expansion of V_{LL} in x as

$$V_{LL} = \frac{\pi^2 \phi^4}{6} [y S_0(yL) + x S_1(yL) + x^2 L S_2(yL) + O(x^3)] \quad (39)$$

it is easy to check that the resulting coefficient functions $S_i(yL)$ ($i=0,1,2$) coincides with the results of Ref.[4] exactly. Finally, let us consider the effect of the multi-mass scales on the prediction of the magnitude of the Higgs mass. For simplicity, we will consider up to two loop order where the difference between the case of single mass scale and that of multi-mass scale appears for the first time. By expanding the RG improved effective potential for the CW model given in Eq.(19) up to two loop order where $\phi_0(t)$ and $y_0(t)$ are given in Eq.(24) and Eq.(35), we obtain

$$V_{LL} = \frac{\pi^2 \phi^4}{6} [y + \frac{3}{8} y^2 L_1 + \frac{1}{24} y^2 L_2 + \frac{9}{2} x^2 L_3 + \frac{9}{64} y^3 L_1^2 + \frac{1}{32} y^3 L_1 L_2 + (-\frac{9}{16} xy^2 + \frac{27}{8} x^2 y) L_1 L_3 + \frac{1}{576} y^3 L_2^2 + (-\frac{1}{16} xy^2 + \frac{3}{8} x^2 y) L_2 L_3 + (\frac{5}{16} xy^2 - \frac{15}{8} x^2 y + \frac{15}{4} x^3) L_3^2] \quad (40)$$

In order to obtain the total RG improved effective potential, one should incorporate the counter-term as

$$V_{tot} = V_{LL} + K \phi^4 \quad (41)$$

where K can be determined by the application of the renormalization condition

$$\left[\frac{d^4 V_{eff}(\phi)}{d\phi^4} \right]_{\phi=\mu} = 4\pi^2 y \quad (42)$$

From Eqs.(40)-(42), we can obtain

$$V_{tot} = \frac{\pi^2 \phi^4}{6} \{ y + (\frac{5}{12} y^2 + \frac{9}{2} x^2)(L - \frac{25}{6}) + (\frac{25}{144} y^3 - \frac{5}{16} xy^2 + \frac{15}{8} x^2 y + \frac{15}{4} x^3)(L^2 - \frac{35}{3}) + [(\frac{5}{16} y^3 - \frac{9}{16} xy^2 + \frac{27}{8} x^2 y)\delta_1 + (\frac{5}{144} y^3 - \frac{1}{16} xy^2 + \frac{3}{8} x^2 y)\delta_2 + \frac{15}{2} x^3 \delta_3](L - \frac{25}{6}) \} \quad (43)$$

where

$$\delta_i \equiv L_i - L \quad (44)$$

The δ_i dependent terms corresponds to the difference between the single mass scale and the multi-mass scales L_i . Then we can obtain the ratio of the scalar field and gauge field as

$$\frac{m_\phi^2}{m_A^2} = \left[\frac{V''_{tot}(\phi)}{e^2 \phi^2} \right]_{\phi=\mu} = r_s + \delta r \quad (45)$$

where r_s contains those terms coming from single mass scale case such as

TABLE I: The difference between the prediction of $\frac{m_\phi^2}{m_A^2}$ in case of single mass scale case (r_s) and that of multi-mass scale case (δr) and absolute value of their ratio r for the typical values parameters x and y

x	y	r_s	δr	$r = \frac{\delta r}{r_s} $
0.01	0.0017	0.00151	0.00137	0.09095
0.01	0.394	4.31415	-5.94373	1.37773
0.03	0.0168	0.04819	0.00151	0.03126
0.03	0.384	1.43934	-1.74417	1.21197
0.05	0.0547	0.09435	-0.01865	0.19769
0.05	0.363	0.71967	-0.90571	1.25850
0.07	0.1536	0.19317	-0.12132	0.62804
0.07	0.2757	0.36282	-0.36215	0.99816

$$r_s = \frac{y}{2x} - \frac{1}{4x} \left(\frac{5}{2}y^2 + 27x^2 \right) - \frac{1}{4x} \left(\frac{275}{72}y^3 - \frac{55}{8}xy^2 + \frac{165}{4}x^2y + \frac{165}{2}x^3 \right) \quad (46)$$

and δr contains those terms coming from multi-mass scale such as

$$\delta r = -\frac{1}{4x} \left[\left(\frac{15}{8}y^3 - \frac{27}{8}xy^2 + \frac{81}{4}x^2y \right) \delta_1 + \left(\frac{5}{24}y^3 - \frac{3}{8}xy^2 + \frac{9}{4}x^2y \right) \delta_2 + 45x^3\delta_3 \right] \quad (47)$$

The relation between x and y can be determined from the condition $[dV_{tot}/d\phi]_{\phi=\mu} = 0$ for single mass scale case and there exist two different values of the coupling constants y for each given values of x [4]. In table I, we give r_s , δr and their ratio for typical values of the coupling constants x and y . As we can see in this table, their ratio becomes important in case of large value of y (strong λ phase) for given x .

III. DISCUSSIONS AND CONCLUSIONS

In this paper, we have obtained the RG improvement of the leading-logarithms of the effective potential for the CW model by using the method of RG improvement in case of the multi-mass scale. Then we have investigated the effect of the multi-mass scale on the prediction of the Higgs boson mass in case of the two-loop order effective potential where the difference between the case of single mass scale and that of multi-mass scale appears for the first time. We have seen that the effect of the multi-mass scale dependent terms on the prediction of the Higgs mass becomes important in case of strong λ phase. This fact implies that when we expand the CW model to the more realistic model, one should take account of the multi-mass scale of the effective potential.

ACKNOWLEDGMENTS

This research was supported by the Institute of the Basic Science Research Center.

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